

ON QUOTIENTS OF RIEMANN ZETA VALUES AT ODD AND EVEN INTEGER ARGUMENTS

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ABSTRACT. We show for even positive integers n that the quotient of the Riemann zeta values $\zeta(n+1)$ and $\zeta(n)$ satisfies the equation

$$\frac{\zeta(n+1)}{\zeta(n)} = \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{2^{n+1}-1}\right) \frac{\mathcal{L}^*(\mathbf{p}_n)}{\mathbf{p}'_n(0)},$$

where $\mathbf{p}_n \in \mathbb{Z}[x]$ is a certain monic polynomial of degree n and $\mathcal{L}^* : \mathbb{C}[x] \rightarrow \mathbb{C}$ is a linear functional, which is connected with a special L -function. There exists the decomposition $\mathbf{p}_n(x) = x(x+1)\mathbf{q}_n(x)$. If $n = p+1$ where p is an odd prime, then \mathbf{q}_n is an Eisenstein polynomial and therefore irreducible over $\mathbb{Z}[x]$.

1. INTRODUCTION

Euler's beautiful formula

$$\zeta(n) = -\frac{1}{2} \frac{(2\pi i)^n}{n!} B_n \quad (n \in 2\mathbb{N}) \quad (1.1)$$

establishes a relationship between the Riemann zeta function

$$\zeta(s) = \sum_{n \geq 1} n^{-s} \quad (s \in \mathbb{C}, \operatorname{Re}(s) > 1)$$

at even positive integer arguments and the Bernoulli numbers B_n defined by

$$\frac{s}{e^s - 1} = \sum_{n \geq 0} B_n \frac{s^n}{n!} \quad (|s| < 2\pi).$$

These numbers are rational where $B_n = 0$ for odd $n > 1$. The functional equation (cf. [9])

$$\zeta(1-s) = 2(2\pi)^{-s} \cos\left(\frac{\pi s}{2}\right) \Gamma(s) \zeta(s) \quad (s \in \mathbb{C}) \quad (1.2)$$

leads to the values at negative integer arguments given by

$$\zeta(1-n) = -\frac{B_n}{n} \quad (n \in \mathbb{N}, n \geq 2), \quad (1.3)$$

which have remarkable p -adic properties. Sequences of these divided Bernoulli numbers in certain arithmetic progressions encode information about zeros of nontrivial p -adic zeta functions; so far only unique simple zeros have been found, see [4] and [7] for the theory.

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It remains an open question whether there exists a closed formula for $\zeta(3), \zeta(5), \zeta(7), \dots$ with “certain” constants in the sense of (1.1). However, our goal is to show some properties of the quotients $\zeta(n+1)/\zeta(n)$ for $n \in 2\mathbb{N}$ relating them to a “logarithmic part”.

Define the linear forward difference operator ∇ and its powers by

$$\nabla^n f(s) = \sum_{\nu=0}^n (-1)^{n-\nu} \binom{n}{\nu} f(s + \nu)$$

for integers $n \geq 0$ and any function $f : \mathbb{C} \rightarrow \mathbb{C}$. We use the expression, for example,

$$\nabla^n f(s+t) \Big|_{s=1}$$

to indicate the variable and an initial value when needed.

In 1930 Hasse [3] constructed a globally convergent series representation of ζ . He further showed that the following series representation of the Dirichlet eta function, derived by Knopp via Euler transformation [6],

$$\eta(s) = (1 - 2^{1-s})\zeta(s) = \sum_{n \geq 0} \frac{(-1)^n}{2^{n+1}} \nabla^n x^{-s} \Big|_{x=1} \quad (1.4)$$

is also valid for all $s \in \mathbb{C}$ and uniformly convergent on any compact subsets. Compared to the first series of Hasse mentioned above, this gives a globally and faster convergent series of ζ except for the set

$$\mathcal{E} = \{1 + 2\pi i n / \log 2 : n \in \mathbb{Z}\}.$$

The derivative of (1.4) leads to the function, see [2] for a wider context,

$$\mathcal{H}(s) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{2^n} \nabla^n x^{-s} \log x \Big|_{x=1} \quad (s \in \mathbb{C}), \quad (1.5)$$

which satisfies

$$\mathcal{H}(s) = 2^{2-s} \zeta(s) \log 2 + 2(1 - 2^{1-s}) \zeta'(s) \quad (s \in \mathbb{C}). \quad (1.6)$$

For now, let $n \in 2\mathbb{N}$. Note that $n \notin \mathcal{E}$. Considering the functional equation (1.2) and using (1.6), one easily sees that

$$\zeta(n+1) = 2 \frac{(2\pi i)^n}{n!} \zeta'(-n) = - \frac{(2\pi i)^n}{n!} \frac{\mathcal{H}(-n)}{2^{n+1} - 1},$$

since $\zeta(-n) = 0$ is a trivial zero by (1.3). By Euler’s formula (1.1) we finally derive that

$$\frac{\zeta(n+1)}{\zeta(n)} = \frac{2}{B_n} \frac{\mathcal{H}(-n)}{2^{n+1} - 1}. \quad (1.7)$$

We will consider a special L -function that is connected with (1.5). Let $o(\cdot)$ be Landau’s little o symbol.

Theorem 1.1. *The L -function*

$$\mathcal{L}(s) = \sum_{n \geq 1} a_n n^{-s} \quad (s \in \mathbb{C})$$

is an entire function on \mathbb{C} where

$$a_n = \frac{(-1)^{n+1}}{2^n} \nabla^n \log x \Big|_{x=1} = o(2^{-n}).$$

More precisely, the values

$$l_n = (-1)^{n+1} \nabla^n \log x \Big|_{x=1} \in (0, \log 2] \quad (n \geq 1)$$

define a strictly decreasing sequence with limit 0. The linear functional

$$\mathcal{L}^* : \mathbb{C}[x] \rightarrow \mathbb{C}, \quad \mathcal{L}^*(f) = \sum_{n \geq 1} a_n f(n),$$

is absolutely convergent for any $f \in \mathbb{C}[x]$. A special value is given by

$$\mathcal{L}(0) = \mathcal{L}^*(1) = \log(\pi/2).$$

Theorem 1.2. *We have the following relations for $n \geq 1$:*

$$\mathcal{H}(-n) = 2^{-n} \mathcal{L}^*(\mathfrak{p}_n)$$

where

$$\begin{aligned} \mathfrak{p}_n(x) &= \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \binom{x+\nu}{\nu} \nabla^\nu (x+1)^n, \\ \mathfrak{p}_1(x) &= x+1, \\ \mathfrak{p}_2(x) &= x(x+1), \\ \mathfrak{p}_n(x) &= \begin{cases} (x+1)^2 \mathfrak{q}_n(x) & \text{if odd } n \geq 3, \\ x(x+1) \mathfrak{q}_n(x) & \text{if even } n \geq 4. \end{cases} \end{aligned}$$

The polynomials $\mathfrak{p}_n, \mathfrak{q}_n \in \mathbb{Z}[x]$ are monic of degrees $n, n-2$, respectively. The value $\mathfrak{p}_n(0)$ is related to the tangent numbers by

$$\mathfrak{p}_n(0) = \tanh^{(n)}(0) = 2^{n+1} (2^{n+1} - 1) \frac{B_{n+1}}{n+1} \quad (n \geq 1).$$

Special values of the derivative are given by

$$\begin{aligned} \mathfrak{p}'_n(-1) &= -\mathfrak{p}_{n-1}(0) & (n \geq 2), \\ \mathfrak{p}'_n(0) &= (n-1) \mathfrak{p}_{n-1}(0) & (n \in 2\mathbb{N}). \end{aligned}$$

Now, we can state our main result. We further use the notations \mathfrak{p}_n and \mathfrak{q}_n below.

Theorem 1.3. *If $n \in 2\mathbb{N}$, then*

$$\frac{\zeta(n+1)}{\zeta(n)} = \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{2^{n+1}-1}\right) \frac{\mathcal{L}^*(\mathfrak{p}_n)}{\mathfrak{p}'_n(0)}. \quad (1.8)$$

There exists the limit

$$\lim_{n \rightarrow \infty \atop 2|n} \frac{\mathcal{L}^*(\mathfrak{p}_n)}{\mathfrak{p}'_n(0)} = 1.$$

Generally, if $\alpha_j \in \mathbb{Q}^\times$ and $n_j \in 2\mathbb{N}$ for $j = 1, \dots, N$ and fixed $N \geq 1$, where the integers n_j are strictly increasing, then

$$\sum_{j=1}^N \alpha_j \zeta(n_j+1)/\zeta(n_j) = \mathcal{L}^*(\mathfrak{p})$$

where $\mathfrak{p} \in \mathbb{Q}[x]$ and $\deg \mathfrak{p} = n_N$.

Table 1.4.

$$\begin{aligned} \zeta(3)/\zeta(2) &= \frac{3}{7} \mathcal{L}^*(\mathfrak{p}_2), \quad \mathfrak{p}_2(x) = x^2 + x, \\ \zeta(5)/\zeta(4) &= -\frac{15}{124} \mathcal{L}^*(\mathfrak{p}_4), \quad \mathfrak{p}_4(x) = x^4 - 2x^3 - 9x^2 - 6x \\ &= (x^2 + x)(x^2 - 3x - 6), \\ \zeta(7)/\zeta(6) &= \frac{21}{2032} \mathcal{L}^*(\mathfrak{p}_6), \quad \mathfrak{p}_6(x) = x^6 - 9x^5 - 15x^4 + 65x^3 + 150x^2 + 80x \\ &= (x^2 + x)(x^4 - 10x^3 - 5x^2 + 70x + 80). \end{aligned}$$

Of course one can simplify (1.8) to

$$\zeta(n+1)/\zeta(n) = (2^{n-1}(2^{n+1}-1)B_n)^{-1} \mathcal{L}^*(\mathfrak{p}_n) \quad (n \in 2\mathbb{N}),$$

but this needs again the definition of the Bernoulli numbers.

Theorem 1.5. *Let p be an odd prime and $n = p + 1$. Then the polynomial \mathfrak{q}_n is an Eisenstein polynomial and consequently irreducible over $\mathbb{Z}[x]$.*

On the basis of some computations, see also Table A.2, we raise the following conjecture.

Conjecture 1.6. *The polynomials \mathfrak{q}_n are irreducible over $\mathbb{Z}[x]$ for all $n \geq 4$.*

2. PRELIMINARIES

We need some lemmas to prove the theorems in the following. For properties of binomial coefficients, Stirling numbers, and finite differences we refer to [1].

The harmonic numbers are defined by

$$\mathbf{H}_n = \sum_{k=1}^n \frac{1}{k} \quad (n \geq 1).$$

The Stirling numbers $\mathbf{S}_1(n, k)$ of the first kind and $\mathbf{S}_2(n, k)$ of the second kind are usually defined by

$$(x)_n = \sum_{k=0}^n \mathbf{S}_1(n, k) x^k \quad (n \geq 0), \quad (2.1)$$

$$x^n = \sum_{k=0}^n \mathbf{S}_2(n, k) (x)_k \quad (n \geq 0), \quad (2.2)$$

where the falling factorials are given by

$$(x)_0 = 1 \quad \text{and} \quad (x)_k = x(x-1) \cdots (x-k+1) \quad \text{for} \quad k \geq 1.$$

We further use the related numbers

$$\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle = \nabla^k x^n \Big|_{x=0} = k! \mathbf{S}_2(n, k), \quad (2.3)$$

which obey the recurrence

$$\left\langle \begin{matrix} n+1 \\ k \end{matrix} \right\rangle = k \left(\left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ k-1 \end{matrix} \right\rangle \right). \quad (2.4)$$

Then (2.2) turns into

$$x^n = \sum_{k=0}^n \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle \binom{x}{k} \quad (n \geq 0). \quad (2.5)$$

Note that $\mathbf{S}_1(n, n) = \mathbf{S}_2(n, n) = \mathbf{S}_2(n, 1) = 1$ and $\mathbf{S}_1(n, 1) = (-1)^{n-1}(n-1)!$ for $n \geq 1$. Further $\mathbf{S}_1(n, 0) = \mathbf{S}_2(n, 0) = \left\langle \begin{matrix} n \\ 0 \end{matrix} \right\rangle = \delta_{n,0}$ for $n \geq 0$ using Kronecker's delta.

Lemma 2.1. *If p is a prime and $k \geq 0$, then*

$$(1 + (x-1)_{p-1})^{(k)} \Big|_{x=0} \equiv -\delta_{k,p-1} \pmod{p}.$$

Proof. The case $p = 2$ is trivial. Let p be an odd prime. By (2.1) we obtain that

$$1 + (x-1)_{p-1} = 1 + (x)_p/x = 1 + \sum_{k=1}^p \mathbf{S}_1(p, k) x^{k-1} \equiv x^{p-1} \pmod{p}$$

using the property ([1, 6.51, p. 314])

$$\mathbf{S}_1(p, k) \equiv 0 \pmod{p} \quad (1 < k < p)$$

and Wilson's theorem to derive that $\mathbf{S}_1(p, 1) = (-1)^{p-1}(p-1)! \equiv -1 \pmod{p}$. Only the $(p-1)$ -th derivative of x^{p-1} at $x = 0$ does not vanish and equals $-1 \pmod{p}$ using Wilson's theorem again. \square

Lemma 2.2. *Let $k, n \geq 1$ be integers. Then*

$$\nabla^k x^n \Big|_{x=1} = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ k+1 \end{matrix} \right\rangle = \frac{1}{k+1} \nabla^{k+1} x^{n+1} \Big|_{x=0}.$$

Proof. This follows by

$$\nabla^k x^n \Big|_{x=1} = \nabla^k x^n \Big|_{x=0} + \nabla^{k+1} x^n \Big|_{x=0}$$

and using (2.3) and (2.4). \square

Lemma 2.3. *If p is a prime and $k \geq 0$, then*

$$\frac{1}{k!} \nabla^k x^p \Big|_{x=1} \equiv d_{p,k} \pmod{p}$$

where $d_{p,k} = 1$ for $k = 0, 1, p$ and $d_{p,k} = 0$ otherwise.

Proof. By Lemma 2.2 and (2.3) we have

$$\tilde{d}_{p,k} := \frac{1}{k!} \nabla^k x^p \Big|_{x=1} = \frac{1}{k!} \left(\left\langle \frac{p}{k} \right\rangle + \left\langle \frac{p}{k+1} \right\rangle \right) = \mathbf{S}_2(p, k) + (k+1) \mathbf{S}_2(p, k+1).$$

It is well-known ([1, p. 314, 6.51]) that

$$\mathbf{S}_2(p, k) \equiv 0 \pmod{p} \quad (1 < k < p).$$

Thus $\tilde{d}_{p,k} \equiv d_{p,k} \pmod{p}$ follows easily. \square

Lemma 2.4 (Leibniz rule). *For integers $n \geq 0$ and any functions $f, g : \mathbb{C} \rightarrow \mathbb{C}$, we have*

$$\nabla^n (f(s)g(s)) = \sum_{\nu=0}^n \binom{n}{\nu} (\nabla^\nu f(s)) (\nabla^{n-\nu} g(s+\nu)).$$

Lemma 2.5. *Let $k, n, r \geq 1$ be integers and $f : \mathbb{N} \rightarrow \mathbb{C}$ be a function. We have*

$$\nabla^n x^r f(x) \Big|_{x=1} = \sum_{\nu=0}^{\min(n,r)} \lambda_{r,\nu}(n) \nabla^{n-\nu} f(x) \Big|_{x=1}$$

with

$$\lambda_{r,\nu}(n) = \sum_{k=\nu}^r \left\langle \frac{r}{k} \right\rangle \binom{n}{\nu} \binom{n+1-\nu}{k-\nu}$$

where $\lambda_{r,\nu} \in \mathbb{Z}[x]$ and $\deg \lambda_{r,\nu} = r$.

Proof. We use Lemma 2.4 to obtain that

$$\nabla^n x^r f(x) \Big|_{x=1} = \sum_{\nu=0}^{\min(n,r)} \binom{n}{\nu} \nabla^\nu (x+n-\nu)^r \Big|_{x=1} \nabla^{n-\nu} f(x) \Big|_{x=1}.$$

Applying ∇^ν to (2.5) provides that

$$\begin{aligned}\lambda_{r,\nu}(n) &= \binom{n}{\nu} \nabla^\nu (x + n - \nu)^r \Big|_{x=1} \\ &= \sum_{k=\nu}^r \langle r \rangle_k \binom{n}{\nu} \binom{n+1-\nu}{k-\nu}\end{aligned}\tag{2.6}$$

$$= \sum_{k=\nu}^r \binom{k}{\nu} \mathbf{S}_2(r, k) (n)_\nu (n+1-\nu)_{k-\nu}.\tag{2.7}$$

The last equation follows by (2.3) and implies that $\lambda_{r,\nu} \in \mathbb{Z}[x]$ and $\deg \lambda_{r,\nu} = r$. \square

Proposition 2.6 ([5, Theorem]). *Define*

$$\mathbf{F}_n(x) = \sum_{\nu=1}^n \left\langle \frac{n}{\nu} \right\rangle x^\nu, \quad \widehat{\mathbf{F}}_n(x) = \sum_{\nu=1}^n \left\langle \frac{n}{\nu} \right\rangle \mathbf{H}_\nu x^\nu \quad (n \geq 1).$$

For $n \in 2\mathbb{N}$ we have the identity

$$\widehat{\mathbf{F}}_n\left(-\frac{1}{2}\right) = -\frac{n-1}{2} \mathbf{F}_{n-1}\left(-\frac{1}{2}\right).$$

Proposition 2.7 (Kummer congruences [7, Theorem 7 (2), p. 44]). *Let p be an odd prime and $n, m \in 2\mathbb{N}$. If $n \equiv m \not\equiv 0 \pmod{p-1}$, then*

$$\frac{B_n}{n} \equiv \frac{B_m}{m} \pmod{p}.$$

3. PROOFS

Proof of Theorem 1.1. Let $n \geq 1$. We use the identity, cf. [8, p. 54],

$$\nabla^n \log x \Big|_{x=a} = (-1)^{n-1} (n-1)! \int_a^{a+1} \frac{dt}{t(t+1) \cdots (t+n-1)}.$$

Hence

$$l_n = (-1)^{n+1} \nabla^n \log x \Big|_{x=1} = \frac{1}{n} \int_0^1 \frac{dt}{(t+1)(\frac{t}{2}+1) \cdots (\frac{t}{n}+1)}.$$

Define $\phi_n(t) = 1 + t/n$ where ϕ_n maps $(0, 1]$ onto $(1, 1 + 1/n]$. The integrand above differs by the factor ϕ_{n+1}^{-1} regarding n and $n+1$. Estimating these integrals we then obtain that

$$nl_n > (n+1)l_{n+1} > nl_{n+1}.$$

Consequently $(l_n)_{n \geq 1}$ defines a positive strictly decreasing sequence

$$\log 2 = l_1 > l_2 > l_3 > \dots$$

with limit $l_\infty = 0$. Thus, the coefficients of $\mathcal{L}(s)$ obey that $a_n = o(2^{-n})$. As usual, write $s = \sigma + it$. The estimate

$$|\mathcal{L}(s)| \leq \sum_{n \geq 1} |a_n n^{-s}| < \sum_{n \geq 1} 2^{-n} n^{-\sigma} < \infty$$

for any $\sigma, t \in \mathbb{R}$ shows that $\mathcal{L}(s)$ is an entire function on \mathbb{C} . Since \mathcal{L}^* is linear and $\mathcal{L}^*(x^m) = \mathcal{L}(-m)$ for $m \in \mathbb{N}_0$, it follows that \mathcal{L}^* is absolutely convergent for any $f \in \mathbb{C}[x]$. The well-known values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$ yield the special value $\mathcal{L}^*(1) = \mathcal{L}(0) = \mathcal{H}(0) = \log(\pi/2)$ concerning (1.5) and (1.6). \square

Proposition 3.1. *We have the following relations for $r \geq 1$:*

$$\mathcal{H}(-r) = 2^{-r} \mathcal{L}^*(\mathfrak{p}_r)$$

where

$$\mathfrak{p}_r(x) = \sum_{\nu=0}^r (-1)^\nu 2^{r-\nu} \lambda_{r,\nu}(x+\nu) \quad (3.1)$$

with $\lambda_{r,\nu}$ as defined in Lemma 2.5. The polynomial $\mathfrak{p}_r \in \mathbb{Z}[x]$ is monic of degree r .

Proof. Let $r \geq 1$ be a fixed integer. By Lemma 2.5 we know that $\mathfrak{p}_r \in \mathbb{Z}[x]$ and $\deg \mathfrak{p}_r \leq r$, since $\lambda_{r,\nu} \in \mathbb{Z}[x]$ and $\deg \lambda_{r,\nu} = r$ for $\nu = 0, \dots, r$. Due to absolute convergence ensured by Theorem 1.1, we derive that

$$\begin{aligned} 2^{-r} \mathcal{L}^*(\mathfrak{p}_r) &= \sum_{\nu=0}^r \sum_{n \geq 1} \frac{(-1)^{n+\nu+1}}{2^{n+\nu}} \lambda_{r,\nu}(n+\nu) \nabla^n \log x \Big|_{x=1} \\ &= \sum_{\nu=0}^r \sum_{n > \nu} \frac{(-1)^{n+1}}{2^n} \lambda_{r,\nu}(n) \nabla^{n-\nu} \log x \Big|_{x=1} \\ &= \sum_{n \geq 1} \frac{(-1)^{n+1}}{2^n} \sum_{\nu=0}^{\min(n,r)} \lambda_{r,\nu}(n) \nabla^{n-\nu} \log x \Big|_{x=1} \\ &= \sum_{n \geq 1} \frac{(-1)^{n+1}}{2^n} \nabla^n x^r \log x \Big|_{x=1} \\ &= \mathcal{H}(-r). \end{aligned}$$

The last two steps follow by Lemma 2.5 and (1.5). Define the linear functional

$$[\cdot]_n : \mathbb{R}[x] \rightarrow \mathbb{R}, \quad f \mapsto \frac{f^{(n)}(0)}{n!},$$

giving the n th coefficient of a polynomial f . By virtue of (2.7) and (3.1) we easily obtain

$$\begin{aligned} [\mathfrak{p}_r(x)]_r &= \left[\sum_{\nu=0}^r (-1)^\nu 2^{r-\nu} \binom{r}{\nu} \mathbf{S}_2(r, r)(x+\nu)_r (x+1)_0 \right]_r \\ &= \sum_{\nu=0}^r \binom{r}{\nu} (-1)^\nu 2^{r-\nu} = (2-1)^r = 1, \end{aligned}$$

since all other terms vanish. This shows that \mathfrak{p}_r is a monic polynomial of degree r . \square

Proposition 3.2. *The polynomials \mathfrak{p}_n have the following properties for $n \geq 1$:*

$$\begin{aligned}\mathfrak{p}_n(x) &= \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \binom{x+\nu}{\nu} \nabla^\nu (x+1)^n, \\ \mathfrak{p}_1(x) &= x+1, \\ \mathfrak{p}_2(x) &= x(x+1), \\ \mathfrak{p}_n(x) &= \begin{cases} (x+1)^2 \mathfrak{q}_n(x) & \text{if odd } n \geq 3, \\ x(x+1) \mathfrak{q}_n(x) & \text{if even } n \geq 4. \end{cases}\end{aligned}$$

The polynomials $\mathfrak{q}_n \in \mathbb{Z}[x]$ are monic of degree $n-2$. Moreover,

$$\mathfrak{p}_n(0) = \tanh^{(n)}(0) = -\mathfrak{p}'_{n+1}(-1) = 2^{n+1}(2^{n+1}-1) \frac{B_{n+1}}{n+1}.$$

Proof. Let $n \geq 1$. From (2.6) and (3.1) we deduce that

$$\begin{aligned}\mathfrak{p}_n(x) &= \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \sum_{k=\nu}^n \langle n \rangle_k \binom{x+\nu}{\nu} \binom{x+1}{k-\nu} \\ &= \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \binom{x+\nu}{\nu} \nabla^\nu (x+1)^n,\end{aligned}\tag{3.2}$$

where the last part follows by (2.5) and applying ∇^ν . Comparing with (1.4) and using (1.3) we instantly derive that

$$\mathfrak{p}_n(0) = \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \nabla^\nu x^n \Big|_{x=1} = -c_{n+1} \zeta(-n) = c_{n+1} \frac{B_{n+1}}{n+1}\tag{3.3}$$

with an additional factor $c_{n+1} = 2^{n+1}(2^{n+1}-1)$. This equals the n th tangent number except for the sign such that $\mathfrak{p}_n(0) = \tanh^{(n)}(0)$, see [1, p. 287]. We further obtain that

$$\mathfrak{p}_n(-1) = \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \binom{\nu-1}{\nu} \nabla^\nu x^n \Big|_{x=0} = 2^n \nabla^0 x^n \Big|_{x=0} = 0.\tag{3.4}$$

Next, we show that

$$\mathfrak{p}'_{n+1}(-1) = -\mathfrak{p}_n(0).\tag{3.5}$$

Since $\mathfrak{p}_{n+1}(-1) = 0$, $x+1$ is a factor of $\mathfrak{p}_{n+1}(x)$. Hence

$$\frac{\mathfrak{p}_{n+1}(x)}{x+1} = 2^{n+1}(x+1)^n + \sum_{\nu=1}^{n+1} (-1)^\nu 2^{n+1-\nu} \frac{1}{\nu} \binom{x+\nu}{\nu-1} \nabla^\nu (x+1)^{n+1}.$$

By L'Hôpital's rule we obtain that

$$\mathfrak{p}'_{n+1}(-1) = \lim_{x \rightarrow -1} \frac{\mathfrak{p}_{n+1}(x)}{x+1} = - \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \frac{1}{\nu+1} \nabla^{\nu+1} x^{n+1} \Big|_{x=0}.$$

In view of Lemma 2.2 and (3.3) we then get (3.5). From (3.3), (3.4), (3.5), and $B_n = 0$ for odd $n \geq 3$, we finally conclude that $x + 1 \mid \mathfrak{p}_n(x)$ for $n \geq 1$, $x \mid \mathfrak{p}_n(x)$ for even $n \geq 2$, and $(x + 1)^2 \mid \mathfrak{p}_n(x)$ for odd $n \geq 3$. The rest follows by the fact that \mathfrak{p}_n is a monic polynomial of degree n . \square

Proposition 3.3. *We have*

$$\mathfrak{p}'_n(0) = (n - 1) \mathfrak{p}_{n-1}(0) \quad (n \in 2\mathbb{N}).$$

Proof. We first observe by (3.3) and Lemma 2.2 that

$$\mathfrak{p}_n(0) = \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} \left(\left\langle \begin{matrix} n \\ \nu \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ \nu+1 \end{matrix} \right\rangle \right) = - \sum_{\nu=1}^n (-1)^\nu 2^{n-\nu} \left\langle \begin{matrix} n \\ \nu \end{matrix} \right\rangle \quad (3.6)$$

for $n \geq 1$. Now, let $n \in 2\mathbb{N}$. Define

$$H_k(x) = \sum_{j=1}^k \frac{1}{x+j} \quad (k \geq 1)$$

where $\mathbf{H}_k = H_k(0)$. Via (3.2) one easily sees that the derivative is given by

$$\mathfrak{p}'_n(x) = 2n \mathfrak{p}_{n-1}(x) + f_n(x)$$

with

$$f_n(x) = \sum_{\nu=1}^n (-1)^\nu 2^{n-\nu} \binom{x+\nu}{\nu} H_\nu(x) \nabla^\nu (x+1)^n.$$

Define

$$h_n = \sum_{\nu=1}^n (-1)^\nu 2^{n-\nu} \mathbf{H}_\nu \left\langle \begin{matrix} n \\ \nu \end{matrix} \right\rangle.$$

Using Lemma 2.2 and (3.3) we then obtain that

$$\begin{aligned} f_n(0) &= \sum_{\nu=1}^n (-1)^\nu 2^{n-\nu} \mathbf{H}_\nu \left(\left\langle \begin{matrix} n \\ \nu \end{matrix} \right\rangle + \left\langle \begin{matrix} n \\ \nu+1 \end{matrix} \right\rangle \right) \\ &= h_n - 2(h_n + \mathfrak{p}_{n-1}(0)) = -h_n - 2\mathfrak{p}_{n-1}(0), \end{aligned}$$

where the second part follows by the substitution $\mathbf{H}_\nu = \mathbf{H}_{\nu+1} - \frac{1}{\nu+1}$ and (3.6). Thus

$$\mathfrak{p}'_n(0) = 2(n-1) \mathfrak{p}_{n-1}(0) - h_n.$$

With the help of Proposition 2.6 we finally achieve that

$$h_n = 2^n \widehat{\mathbf{F}}_n(-1/2) = -2^{n-1} (n-1) \mathbf{F}_{n-1}(-1/2) = (n-1) \mathfrak{p}_{n-1}(0).$$

The last equation follows by (3.6) and shows the result. \square

Proof of Theorem 1.2. The proof jointly follows by Propositions 3.1, 3.2, and 3.3. \square

Proof of Theorem 1.3. Let $n \in 2\mathbb{N}$. By (1.7) and Proposition 3.1 we have

$$\zeta(n+1)/\zeta(n) = (2^{n-1}(2^{n+1}-1)B_n)^{-1} \mathcal{L}^*(\mathfrak{p}_n). \quad (3.7)$$

From Propositions 3.2 and 3.3 we obtain that

$$\mathfrak{p}'_n(0) = \frac{n-1}{n} 2^n (2^n - 1) B_n. \quad (3.8)$$

Plugging (3.8) into (3.7) yields that

$$\frac{\zeta(n+1)}{\zeta(n)} = \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{2^{n+1}-1}\right) \frac{\mathcal{L}^*(\mathfrak{p}_n)}{\mathfrak{p}'_n(0)}.$$

Since $\zeta(n) \rightarrow 1$, $1 - 1/n \rightarrow 1$, and $1 - 1/(2^{n+1}-1) \rightarrow 1$ for $n \rightarrow \infty$, we derive that

$$\lim_{\substack{n \rightarrow \infty \\ 2|n}} \frac{\mathcal{L}^*(\mathfrak{p}_n)}{\mathfrak{p}'_n(0)} = 1.$$

It remains the second part. Since \mathcal{L}^* is linear, we obtain by (3.7) that

$$\alpha \zeta(n+1)/\zeta(n) = \mathcal{L}^*(\hat{\mathfrak{p}}) \quad (3.9)$$

where $\alpha \in \mathbb{Q}^\times$ and $\hat{\mathfrak{p}} \in \mathbb{Q}[x]$ with $\deg \hat{\mathfrak{p}} = n$. Thus, the claimed sum, consisting of terms as in (3.9), follows by the assumption that the integers $n_j \in 2\mathbb{N}$ are strictly increasing. \square

Lemma 3.4. *Let $n > k \geq 1$ be integers. The k th derivative of \mathfrak{p}_n is given by*

$$\mathfrak{p}_n^{(k)}(x) = \sum_{j=0}^k \binom{k}{j} (n-1)_j \sum_{\nu=0}^{n-1-j} (-1)^\nu 2^{n-1-\nu} (x+\nu+1)_{\nu+1}^{(k-j)} \frac{1}{\nu!} \nabla^\nu (x+1)^{n-1-j}.$$

Proof. By Lemma 2.4 we have the identity

$$\nabla^\nu (x+1)^n = \nu \nabla^{\nu-1} (x+1)^{n-1} + (x+\nu+1) \nabla^\nu (x+1)^{n-1} \quad (\nu \geq 1).$$

From (3.2) we then derive that

$$\begin{aligned} \mathfrak{p}_n(x) &= \sum_{\nu=0}^n (-1)^\nu 2^{n-\nu} (x+\nu)_\nu \frac{1}{\nu!} \nabla^\nu (x+1)^n \\ &= \sum_{\nu=0}^{n-1} (-1)^\nu 2^{n-1-\nu} (x+\nu+1)_{\nu+1} \frac{1}{\nu!} \nabla^\nu (x+1)^{n-1} \end{aligned}$$

using the above identity and after some index shifting. The k th derivative follows by the Leibniz rule applied to the terms $(x+\nu+1)_{\nu+1}$ and $\nabla^\nu (x+1)^{n-1}$. \square

Proposition 3.5. *If $n = p + 1$ with p an odd prime, then*

$$\mathfrak{p}_n^{(k)}(0) \equiv 0 \pmod{p} \quad (1 \leq k \leq n-2).$$

Proof. Since $n - 1 = p$, we obtain by Lemma 3.4 that

$$\begin{aligned} \mathfrak{p}_n^{(k)}(0) &\equiv \sum_{\nu=0}^{n-1} (-1)^\nu 2^{n-1-\nu} (x + \nu + 1)_{\nu+1}^{(k)} \Big|_{x=0} \frac{1}{\nu!} \nabla^\nu x^{n-1} \Big|_{x=1} \\ &\equiv \sum_{\nu \in \{0,1,p\}} (-1)^\nu 2^{n-1-\nu} (x + \nu + 1)_{\nu+1}^{(k)} \Big|_{x=0} \\ &\equiv (2(x+1) - (x+2)_2 - (x+p+1)_{p+1})^{(k)} \Big|_{x=0} \pmod{p} \end{aligned}$$

applying Lemma 2.3 and Fermat's little theorem. Using the identities

$$-(x^2 + x) = 2(x+1) - (x+2)_2,$$

$$(x+p+1)_{p+1} \equiv (x+p-1)_{p-1}(x+p+1)_2 \equiv (x-1)_{p-1}(x^2+x) \pmod{p},$$

we achieve that

$$\begin{aligned} \mathfrak{p}_n^{(k)}(0) &\equiv -(A(x)B(x))^{(k)} \Big|_{x=0} \\ &\equiv -\sum_{j=1}^k \binom{k}{j} A(x)^{(j)} B(x)^{(k-j)} \Big|_{x=0} \pmod{p} \end{aligned}$$

where

$$A(x) = x^2 + x, \quad B(x) = 1 + (x-1)_{p-1}.$$

By Lemma 2.1

$$B(x)^{(k-j)} \Big|_{x=0} \equiv -\delta_{k-j,p-1} \pmod{p}.$$

Since $j \geq 1$ and $k - j < n - 2 = p - 1$, it follows that $\mathfrak{p}_n^{(k)}(0) \equiv 0 \pmod{p}$. \square

Proof of Theorem 1.5. Let p be an odd prime and $n = p + 1$. By Proposition 3.2 we have the decomposition $\mathfrak{p}_n(x) = x(x+1)\mathfrak{q}_n(x)$ with

$$\begin{aligned} \mathfrak{p}_n(x) &= x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1x, \\ \mathfrak{q}_n(x) &= x^{n-2} + \beta_{n-3}x^{n-3} + \cdots + \beta_0. \end{aligned}$$

Hence, we have the relations

$$\begin{aligned} \beta_0 &= \alpha_1, \\ \beta_1 &= \alpha_2 - \beta_0, \\ &\dots \\ \beta_{n-3} &= \alpha_{n-2} - \beta_{n-4}. \end{aligned}$$

Since $n - 2 < p$, we obtain by Proposition 3.5 that

$$\alpha_\nu \equiv \mathfrak{p}_n^{(\nu)}(0)/\nu! \equiv 0 \pmod{p} \quad (1 \leq \nu \leq n-2).$$

It easily follows by induction that $p \mid \alpha_\nu$ for $1 \leq \nu \leq n-2$ implies that $p \mid \beta_\nu$ for $0 \leq \nu \leq n-3$. Moreover, we derive by (3.8) that

$$\beta_0 = \alpha_1 = \mathfrak{p}'_n(0) = (n-1)2^n(2^n-1)B_n/n.$$

Now, we have to show that $\text{ord}_p \beta_0 = 1$. It is well-known that $B_2 = 1/6$ and $B_4 = -1/30$. For $p = 3$ we compute $\beta_0 = -6$. Let $p \geq 5$. By Kummer congruences via Proposition 2.7 and Fermat's little theorem we conclude that

$$2^n(2^n - 1)B_n/n \equiv 2^2(2^2 - 1)B_2/2 \equiv 1 \pmod{p}$$

and further that $\text{ord}_p \beta_0 = 1$, since $p = n - 1$. Altogether, this shows that \mathfrak{q}_n is an Eisenstein polynomial and therefore \mathfrak{q}_n is irreducible over $\mathbb{Z}[x]$. \square

APPENDIX A. COMPUTATIONS

Table A.1. Polynomials \mathfrak{p}_n :

$$\mathfrak{p}_1(x) = x + 1.$$

$$\mathfrak{p}_2(x) = x^2 + x.$$

$$\mathfrak{p}_3(x) = x^3 - 3x - 2.$$

$$\mathfrak{p}_4(x) = x^4 - 2x^3 - 9x^2 - 6x.$$

$$\mathfrak{p}_5(x) = x^5 - 5x^4 - 15x^3 + 5x^2 + 30x + 16.$$

$$\mathfrak{p}_6(x) = x^6 - 9x^5 - 15x^4 + 65x^3 + 150x^2 + 80x.$$

$$\mathfrak{p}_7(x) = x^7 - 14x^6 + 210x^4 + 315x^3 - 196x^2 - 588x - 272.$$

$$\mathfrak{p}_8(x) = x^8 - 20x^7 + 42x^6 + 448x^5 + 105x^4 - 2492x^3 - 4116x^2 - 1904x.$$

Table A.2. Polynomials \mathfrak{q}_n :

$$\mathfrak{q}_3(x) = x - 2.$$

$$\mathfrak{q}_4(x) = x^2 - 3x - 6.$$

$$\mathfrak{q}_5(x) = x^3 - 7x^2 - 2x + 16.$$

$$\mathfrak{q}_6(x) = x^4 - 10x^3 - 5x^2 + 70x + 80.$$

$$\mathfrak{q}_7(x) = x^5 - 16x^4 + 31x^3 + 164x^2 - 44x - 272.$$

$$\mathfrak{q}_8(x) = x^6 - 21x^5 + 63x^4 + 385x^3 - 280x^2 - 2212x - 1904.$$

$$\mathfrak{q}_9(x) = x^7 - 29x^6 + 183x^5 + 377x^4 - 2512x^3 - 5076x^2 + 3088x + 7936.$$

$$\mathfrak{q}_{10}(x) = x^8 - 36x^7 + 306x^6 + 504x^5 - 7119x^4 - 15204x^3 + 27804x^2 + 99216x + 71424.$$

The polynomials $\mathfrak{q}_4, \dots, \mathfrak{q}_{10}$ are irreducible over $\mathbb{Z}[x]$.

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